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ON PARTIAL TILTING MODULES AND BOUNDED COMPLEXES

BY

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ABSTRACT

We investigate reasonably large modules with a tilting-type behaviour, regarded as bounded complexes of projective modules.

Introduction

In this paper we compare the behaviour of tilting modules with that of big and functorial "approximations" of tilting modules, regarded as objects of the huge category of right bounded complexes of projective modules.

The motivation behind this note was a question posed by F. Mantese and A. Tonolo shortly before and during the Cotilting Workshop held in Prague in January 2004. Before we state it in a precise form, we may describe, as follows, the two complexes (of projective modules), say $\dot{T}$ and $\dot{C}$, involved in the question. First of all, the bounded complex $\dot{T}$ is the projective resolution of a "selforthogonal" module T (of finite projective dimension) with the following property

- If M is a non-zero module and $\dot{M}$ is its projetcive resolution, then there is a morphism from $\dot{T}$ to a shift of $\dot{M}$ which is not homotopic to zero.

Secondly, $\dot{C}$ is a non-zero right bounded complex with the following property

- Every morphism from $\dot{T}$ to any shift of $\dot{C}$ is homotopic to zero.

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Since $\dot{T}$ is a very special "properly partial tilting" complex, the existence of such a $\dot{C}$ follows from the theory of tilting complexes, introduced by Rickard [Rk]. The question of Mantese and Tonolo is whether such a $\dot{C}$ would be necessarily unbounded. We will show that $\dot{C}$ could be not only bounded, but also that it could be quite small. Surprisingly enough, we may obtain such a complex $\dot{C}$ by "approximating" a direct summand $\dot{B}$ (indecomposable as well as decomposable) of a "properly partial tilting" complex, admitting $\dot{T}$ as a direct summand (Remark 6). We also note that many useful tilting-type complexes seem to inherit from tilting-type modules a "combinatorial nature" [R2, p. 123]. As we shall see, we try to use as many indecomposable complexes as possible, with as many indecomposable components as possible. (Examples A, B and D). In other words, following the terminology of [Sc-ZI], "elementary" complexes will play a big role in the sequel. However, in some sense, also indecomposable complexes with decomposable components (Example C) behave like rather small building blocks.

This paper is organized as follows. Section 1 contains definitions and conventions. Section 2 describes more precisely the different strategies used to construct our bounded complexes. Finally, Section 3 contains examples and proofs.

I would like to thank Professor Mary Schaps, for her immediate and very helpful answer [Sc] to my question on maps "of minimal degree", that is with the largest possible image.

1. Preliminaries

Let R be a ring. We denote by R-Mod the category of all left R–modules. If $M \in R$-Mod, then we write $\operatorname{Add} M$ for the class of all modules isomorphic to direct summands of direct sums of copies of M. Next, for every cardinal λ, we write $M^{(\lambda)}$ for the direct sum of λ copies of M. Finally, we write $M^{\perp_\infty}$ for the class

$$M^{\perp_\infty} = \left\{ X \in R - \text{Mod} : \operatorname{Ext}^i_R(M, X) = 0 \text{ for all } i \geq 1 \right\}.$$

The symbol $\operatorname{pdim}(M)$ denotes the projective dimension of M.

We shall say that an R-module T is a **partial n-tilting module** if the following conditions hold:

- $\operatorname{pdim}(T) \leq n$;
- $\operatorname{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ.

Given a partial n-tilting module T, we shall say that T is an **n-tilting module** if there is a long exact sequence of the form

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0,$$

where $T_i \in \operatorname{Add} T$ for every $i = 0, 1, \ldots, n$.

From now on, we shall say, for brevity, that a partial n-tilting module T is a **large partial n-tilting module** if

$$\operatorname{Ker} \operatorname{Hom}(T, -) \cap T^{\perp_\infty} = 0.$$

Consequently, for every large partial n-tilting module T, we have

$(\star)$ $\quad\operatorname{Hom}(T, I) \neq 0$ for every non-zero injective module I.

If there are only finitely many simple modules and T has finite length, this implies that T is **sincere** [AuReS], that is every simple module appears as a composition factor of T.

Maintaining the terminology introduced above, we recall some properties of partial n-tilting modules.

- For every $n \geq 1$, every n-tilting module T is a large partial n-tilting module [B, p. 371].
- A finitely presented module T is a 1-tilting module if and only if T is a large partial 1-tilting module [C, Theorem 1] (also see [CbF, Theorem 3.2.1 and Section 3.1]).
- If P is a partial 1-tilting module which is not a 1-tilting module, then its *perpendicular class* [CTT], that is the class $P^+ = \operatorname{Ker} \operatorname{Hom}(P, -) \cap P^{\perp_\infty}$, is equivalent to the category of all modules over a suitable endomorphism ring ([CTT, Propositions 1.4 and 1.5] and [GL, Proposition 3.8]).
- For every $n \geq 2$, there exists non faithful large partial n-tilting modules of projective dimension n and Loewy length 2 [D2, Example 4].

As it turns out, some "local" properties plus a "global" and functorial Hom-Ext condition characterize several tilting objects, for instance, tilting complexes in the sense of Rickard [Rk], and tilting objects in the sense of Happel, Reiten and Smalø [HReS]. Let us recall some definitions and conventions.

Given a ring R, we write $K(R)$ for the category of complexes over R with morphisms modulo homotopy. Let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R–modules. Assume the following conditions hold:

(1) $\operatorname{Hom}_{K(R)}(\dot{T}, \dot{T}[i]) = 0$ for every $i \neq 0$,

(2) For every non-zero right bounded complex $\dot{X} \in K(R)$ of projective R–modules there exists some $i \in \mathbb{Z}$, such that $\mathrm{Hom}_{K(R)}(\dot{T}, \dot{X}[i]) \neq 0$.

Then $\dot{T}$ is a **tilting complex**, in the sense of Rickard [Rk] as observed by Miyachi [Mi, condition (iii)', p. 184]. In other words, the global, but functorial, condition (2) can replace a global non-functorial condition on triangulated categories, which says the following

- $\mathrm{add}\,\dot{T}$, the additive category of direct summands of finite direct sums of copies of $\dot{T}$, generates (as a triangulated category) the category of all bounded complexes of finitely generated projective R-modules.

Consequently, given a noetherian ring R, every tilting complex $\dot{T}$ satisfies the following condition ([Sc-ZI, condition (2) in the Definition of p. 190])

- If P is an indecomposable projective module and $\dot{P}$ is the stalk complex $0 \longrightarrow P \longrightarrow 0$, with P in degree 0, then $\dot{P}$ belongs to $\mathrm{add}\,\dot{T}$.

Finally, let $\dot{T} \in K(R)$ be a bounded complex of finitely generated projective R-modules. Then, following the definition of [Sc-ZI] for complexes over noetherian rings, we shall say that $\dot{T}$ is a **partial tilting complex** if $\mathrm{Hom}_{K(R)}(\dot{T}, \dot{T}[i]) = 0$ for every $i \neq 0$.

Throughout the paper, given a module M, the symbol $\dot{M}$ denotes a right bounded complex of projective modules of the form $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, where P_0 is in degree 0 and $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is a fixed projective resolution of M.

Next, K always denotes an algebraically closed field, and we always identify modules with their isomorphism classes (resp., complexes with their homotopy classes). Moreover, if Λ is a K-algebra given by a quiver and relations, according to [R1], then we often replace indecomposable finite dimensional modules by some obvious pictures, describing their composition factors. If M is a module of finite length of the form $\bigoplus_{i=1}^{m} M_i^{d_i}$, where $d_i > 0$ for every i and $M_1, \ldots, M_m$ are pairwise indecomposable, then we denote m by $\delta(M)$. Over a representation-finite algebra given by a quiver, we often denote by x the simple module $S(x)$ corresponding to the vertex x, and by $P(x)$ the projective cover of $S(x)$.

When dealing with a complex $\dot{C}$, we often write Z_i, B_i and H_i instead of $Z_i(\dot{C})$, $B_i(\dot{C})$ and $H_i(\dot{C})$. Moreover, we often write d, instead of d_i, for the usual morphism $C_i \longrightarrow C_{i-1}$.

We end with the conventions used to describe morphisms between indecomposable projective modules P and Q (defined over the K-algebra Λ) with the following useful combinatorial property: the K-dimension of the vector space

$V = \{f \in \mathrm{Hom}_\Lambda(P,Q) : f(P) \neq Q\}$ is at most one. First of all, the symbol $P \longrightarrow Q$ will denote a fixed generator v of V, and the symbol $P \xrightarrow{a} Q$ will denote the morphism av for all $a \in K$. Secondly, we shall use Greek letters $\alpha, \beta, \ldots$ to denote arbitrary morphisms. Finally, the symbols $M \xrightarrow{\simeq} M$ and $M \xrightarrow{0} N$ will denote the identity and the zero morphism.

2. Strategies

We can now state as follows the question discussed in this paper.

Question 1 (F. Mantese and A. Tonolo): *Let T be a large partial n-tilting module which is not an n-tilting module, and let $\dot{T}$ be the projective resolution of T. Finally, let $\dot{C}$ be a non-zero right bounded complex of projective modules, such that every morphism $\dot{T} \longrightarrow \dot{C}[i]$ is homotopic to zero for every $i \in \mathbb{Z}$. Under all these hypotheses, is $\dot{C}$ necessarily unbounded?*

We will give a negative answer to this question by means of rather small complexes $\dot{C}$, quite similar to the projective resolutions $\dot{B}$ of suitable injective module B (of finite projective dimension). To do this, we shall use two lemmas on indecomposable complexes of opposite type, with either one (Lemma 1) or exactly $n+1$ (Lemma 2) non-zero and indecomposable components. We may describe the strategies used in the sequel to obtain $\dot{C}$ from $\dot{B}$ as follows.

- **Left cancellation**, that is cancellation of some of the first non-zero components of $\dot{B}$ on the left hand side (Example A(ii) with n even.).
- **Left and right cancellation**, that is cancellation of some non-zero components of $\dot{B}$ on both sides (Examples A(ii) with n odd).
- **Right addition**, that is addition of at least one new non-zero component to the right hand side of $\dot{B}$ (Example D(ii)).
- **Left perturbation**, that is substitution of the first non-zero component by a proper submodule (Example C(iii)).

More generally, indecomposable complexes $\dot{C}$ of the form $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$, with P and Q indecomposable projective and $P \neq Q$, play an important role, even when the projective dimension of T is very large (Example A). In other words, we could repeat the remarks of Schaps and Zakay–Illouz [Sc-ZI] on the importance of indecomposable complexes "of a special combinatorial character," for instance "two-restricted" and "elementary" [Sc-ZI, Abstract, p. 187; Definition, p. 190].

Moreover, a certain useful complex $\dot{C}$, with more than two non-zero terms (Example C), resembles the examples of "many other indecomposable complexes from which tilting complexes could be constructed" ([Sc-ZI, Note, p. 191]). We now point out two "concealed" combinatorial properties satisfied by the K-algebras Λ considered in the following:

- Many indecomposable modules P are **bricks** [R1], that is we have $\mathrm{End}_\Lambda(P) \simeq K$.
- Every non-zero and non-surjective morphism between indecomposable projective modules is uniquely determined up to a scalar.

Finally, we stress the fact that all four operations described above lead to indecomposable complexes (Proposition 3) with the property that many combinatorial, more or less "visible", data take their smallest possible value.

3. Examples and Proofs

We begin with a lemma on stalk complexes.

Lemma 1: *Let P be an indecomposable projective R-module, let $\dot{C} \in K(R)$ and let $H = \bigoplus_{i \in \mathbb{Z}} H_i$, where H_i is the i-th homology module of $\dot{C}$ for any i. Then the following conditions are equivalent:*
(a) $\mathrm{Hom}_{K(R)}(\dot{P}, \dot{C}[i]) = 0$ *for every* i;
(b) $\mathrm{Hom}_R(P, H) = 0$.

Proof. (a)⇒(b) Assume (a) holds, and $\alpha : P \longrightarrow H_l$ is a morphism for some l. Since P is projective, we may lift α to a morphism $\beta : P \longrightarrow Z_l$. Let $i : Z_l \longrightarrow C_l$ be the canonical inclusion, and let $\gamma = i \circ \beta$. Then the following picture

$$(\star)\qquad \begin{array}{ccccccccc} & & 0 & \longrightarrow & P & \longrightarrow & 0 & & \\ & & & & \big\downarrow{\scriptstyle\gamma} & & & & \\ \cdots & \longrightarrow & C_{l+1} & \longrightarrow & C_l & \longrightarrow & C_{l-1} & \longrightarrow & \cdots \end{array}$$

describes an element $\dot{\gamma} \in \mathrm{Hom}_{K(R)}(\dot{P}, \dot{C}[l])$. On the other hand, by (a), we have $\dot{\gamma} = 0$. Consequently we get $\gamma(P) \subseteq B_l$, and so $\alpha = 0$.

(b)⇒(a) Suppose that (b) holds and that $(\star)$ describes an element $\dot{\gamma} \in \mathrm{Hom}_{K(R)}(\dot{P}, \dot{C}[l])$. Since $\gamma(P) \subseteq Z_l$, we deduce from (b) that $\gamma(P) \subseteq B_l$. Therefore $\dot{\gamma} = 0$, and the lemma is proved. ∎

As was mentioned in the introduction, indecomposable complexes with as few as possible morphisms between two of their non-zero component play an important role in the sequel. In particular, we shall often use the following lemma on complexes over a K-algebra Λ.

Lemma 2: *Let M be an indecomposable non projective Λ-module over a K-algebra Λ. Assume $\dot{M}$ is of the form $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, where $n \geq 2$ and the following condition holds:*

(+) $\mathrm{Hom}_\Lambda(P_i, P_j) \simeq K$ *if either $i = j$ or $i = j+1$, and* $\mathrm{Hom}_\Lambda(P_i, P_j) = 0$ *otherwise.*

Let $\dot{X}$ denote a complex of the form $0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_s \longrightarrow 0$, where $n > r > s \geq 0$ and $\dot{X}$ is obtained from $\dot{M}$ by deleting $n - r \geq 1$ components on the left and $s \geq 0$ components on the right. Then we have $\mathrm{Hom}_{K(\Lambda)}(\dot{M}, \dot{X}[i]) = 0$ for every $i \in \mathbb{Z}$.

Proof. Let $\dot{\alpha}$ be a morphism of the form

$$(1)\qquad \begin{array}{ccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow \cdots \longrightarrow & P_r & \longrightarrow \cdots \longrightarrow & P_s & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha_r} & & \downarrow{\scriptstyle \alpha_s} & & & & \\ & & & 0 \longrightarrow & P_r & \longrightarrow \cdots \longrightarrow & P_s & \longrightarrow & 0 & & \end{array}$$

Since $n > r$, it follows that $\alpha_i = 0$ for every i. Hence we have $\dot{\alpha} = 0$. Next, let $\dot{\beta}$ be a morphism of the form

$$(2)\qquad \begin{array}{ccccccccc} \cdots \longrightarrow & P_{r+1} & \xrightarrow{d_r} & P_r & \longrightarrow \cdots \longrightarrow & P_{s+1} & \xrightarrow{d_s} & P_s & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \beta_r} & & \downarrow{\scriptstyle \beta_{r-1}} & & \downarrow{\scriptstyle \beta_s} & & & \\ 0 \longrightarrow & P_r & \xrightarrow[d_{r-1}]{} & P_{r-1} & \longrightarrow \cdots \xrightarrow[d_s]{} & P_s & \longrightarrow & 0 & \end{array}$$

Since (+) holds, we have $\mathrm{Hom}_R(P_{i+1}, P_i) = \mathrm{End}_R(P_i) \circ d_i$ for every $i = s, \ldots, r$. Consequently, proceeding from left to right, we can find endomorphisms $\gamma_r, \ldots, \gamma_s$ of $P_r, \ldots, P_s$ such that $\gamma_r \circ d_r = \beta_r$, $\gamma_{r-1} \circ d_{r-1} = \beta_{r-1} - d_{r-1} \circ \gamma_r, \ldots, \gamma_s \circ d_s = \beta_s - d_s \circ \gamma_{s+1}$. Hence we have $\dot{\beta} = 0$. Since (+) holds, this implies that $\mathrm{Hom}_{K(\Lambda)}(\dot{M}, \dot{X}[i]) = 0$ for every i, as desired. ∎

We begin with an example where all modules have **Loewy length** ([AF] or [AuReS]) at most equal to two.

Example A (Left cancellation or left and right cancellation): For any $n \geq 2$, there is a large partial n-tilting Λ-module T with the following properties:

(i) T is not faithful and $\operatorname{pdim}(T) = n$.

(ii) There is an indecomposable complex $\dot{X}$ of the form $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$, with P and Q indecomposable projective, such that $\operatorname{Hom}_{K(\Lambda)}(\dot{T}, \dot{X}[i]) = 0$ for every $i \in \mathbb{Z}$.

(iii) Up to shift, the number of choices of $\dot{X}$ in (ii) is $n/2$ if n is even, and $(n-1)/2$ if n is odd.

(iv) Up to shift, $\delta(\Lambda) - \delta(T)$ is at most equal to the number of indecomposable complexes $\dot{Y}$ such that every non-zero component of $\dot{Y}$ is an indecomposable projective module, and $\operatorname{Hom}_{K(\Lambda)}(\dot{T}, \dot{Y}[i]) = 0$ for every $i \in \mathbb{Z}$. Moreover, $\delta(\Lambda) - \delta(T)$ is equal to this number if and only if $\delta(\Lambda) - \delta(T) = 1$, that is if and only if $n = 2, 3$.

(v) The complexes $\dot{X}$ in (ii) and $\dot{Y}$ in (iii) are complexes $\dot{C}$ such that

$$\operatorname{Hom}_{K(\Lambda)}(\dot{C}, \dot{C}[j]) = 0$$

for every integer $j \neq 0$.

Construction. Let Λ be the K-algebra given by the quiver

$$\underset{1}{\bullet} \xrightarrow{a_1} \underset{2}{\bullet} \xrightarrow{a_2} \underset{3}{\bullet} \cdots\cdots\cdots \underset{n}{\bullet} \xrightarrow{a_n} \underset{n+1}{\bullet}$$

with relations $a_{i+1}a_i = 0$ for $i = 1, \ldots, n-1$. Next, let T denote the following injective module:

$$T = \begin{cases} \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 1 & \text{if } n \text{ is even} \\ \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1 & \text{if } n \text{ is odd} \end{cases}$$

Then one easily sees [D2, Example 4 and Remark 5] that T is a large partial n-tilting module, and (i) clearly holds. To end the proof, we first note that $\dot{1}$ is a complex of the form

$$0 \longrightarrow n+1 \longrightarrow \begin{matrix} n \\ n+1 \end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow 0.$$

Next, let $\dot{X}$ denote an indecomposable complex of the form

$$0 \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow 0$$

if n is even, and of the form

$$0 \longrightarrow \begin{matrix} 3 \\ 4 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow 0$$

if n is odd. By Lemma 2, the definition of $\dot{T}$ and $\dot{X}$ implies that

$$\text{(1)} \qquad \mathrm{Hom}_{K(\Lambda)}(\dot{1}, \dot{X}[i]) = 0 \text{ for every } i.$$

We also observe that

(2) An indecomposable projective-injective module $P(j)$ is a summand of T if and only if either j and n are even, or j and n are odd.

This remark and Lemma 1 guarantee that

(3) $\mathrm{Hom}_{K(\Lambda)}(\dot{V}, \dot{X}[i]) = 0$ for every i and every projective summand V of T.

Putting (1) and (3) together, we conclude that $\dot{X}$ satisfies (ii). Moreover, by (2) and Lemmas 1 and 2, we may replace our choice of $\dot{X}$ with that of any indecomposable complex of the form $0 \longrightarrow {\scriptstyle \begin{matrix} j \\ j+1 \end{matrix}} \longrightarrow {\scriptstyle \begin{matrix} j-1 \\ j \end{matrix}} \longrightarrow 0$, where $n \geq j$ and j is even if n is even, while j is odd if n is odd. Consequently, (iii) follows from Lemma 1 and (2). On the other hand, the first part of (iv) follows from the remark that $\delta(\Lambda) - \delta(T)$ coincides with the number of choices of $\dot{X}$ in (iii). Assume first $n = 2, 3$. Then Lemma 1 implies that only a complex of the form $0 \longrightarrow {\scriptstyle \begin{matrix} 2 \\ 3 \end{matrix}} \longrightarrow {\scriptstyle \begin{matrix} 1 \\ 2 \end{matrix}} \longrightarrow 0$ (resp., $0 \longrightarrow {\scriptstyle \begin{matrix} 3 \\ 4 \end{matrix}} \longrightarrow {\scriptstyle \begin{matrix} 2 \\ 3 \end{matrix}} \longrightarrow 0$) satisfies (iv) if $n = 2$ (resp., $n = 3$). Suppose now $n \geq 4$. Then any indecomposable complex obtained "by gluing together" at least two suitable complexes $\dot{X}_1$ and $\dot{X}_2$ satisfying (iii), actually satisfies (iv). For instance, by Lemmas 1 and 2, we may choose the complex

$$0 \longrightarrow \begin{matrix} 4 \\ 5 \end{matrix} \longrightarrow \begin{matrix} 3 \\ 4 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow 0$$

if n is even, and the complex

$$0 \longrightarrow \begin{matrix} 5 \\ 6 \end{matrix} \longrightarrow \begin{matrix} 4 \\ 5 \end{matrix} \longrightarrow \begin{matrix} 3 \\ 4 \end{matrix} \longrightarrow \begin{matrix} 2 \\ 3 \end{matrix} \longrightarrow 0$$

if n is odd. Therefore (iv) holds. Finally, for any $i = 2, \ldots, n$ and $j < i$, any morphism of the form

$$\text{(4)} \qquad \begin{array}{ccccccccccc} & & 0 & \longrightarrow & {\scriptstyle \begin{matrix} i \\ i+1 \end{matrix}} & \longrightarrow \cdots \longrightarrow & {\scriptstyle \begin{matrix} j+1 \\ j+2 \end{matrix}} & \longrightarrow & {\scriptstyle \begin{matrix} j \\ j+1 \end{matrix}} & \longrightarrow & 0 \\ & & & & \Big\downarrow {\scriptstyle \alpha_i} & & \Big\downarrow {\scriptstyle \alpha_{j+1}} & & & & \\ 0 & \longrightarrow & {\scriptstyle \begin{matrix} i \\ i+1 \end{matrix}} & \longrightarrow & {\scriptstyle \begin{matrix} i-1 \\ i \end{matrix}} & \longrightarrow \cdots \longrightarrow & {\scriptstyle \begin{matrix} j \\ j+1 \end{matrix}} & \longrightarrow & 0 & & \end{array}$$

is homotopic to zero. By (ii) and (iii), this implies that also (v) holds. ∎

In the next example we deal with an algebra of global dimension two, such that every indecomposable module is uniserial.

Example B (Left perturbation): There are a large partial 2-tilting Λ-module T and an indecomposable complex $\dot{X}$ with the following properties:

(i) $\mathrm{Hom}_{K(\Lambda)}(\dot{T}, \dot{X}[i]) = 0$ for every $i \in \mathbb{Z}$;

(ii) $\mathrm{Hom}_{K(\Lambda)}(\dot{X}, \dot{X}[j]) = 0$ for every $j \neq 0$;

(iii) $\dot{X}$ is of the form $0 \longrightarrow P \longrightarrow Q \longrightarrow R \longrightarrow 0$ with P, Q, R indecomposable projective modules.

(iv) No indecomposable complex of the form $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$ with P and Q indecomposable projective satisfies (i).

Construction. Let Λ be the K-algebra given by the quiver

$$\underset{1}{\bullet} \xrightarrow{\ a\ } \underset{2}{\bullet} \xrightarrow{\ b\ } \underset{3}{\bullet} \xrightarrow{\ c\ } \underset{4}{\bullet}$$

with relation $ba = 0$. Next, let T be the injective module $\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus\ 1$. Since the global dimension of Λ is equal to 2, the conclusion that T is a partial 2-tilting module follows from a direct check or from [D1, Lemma 1, (ii)]. We also note that the indecomposable modules belonging to $\mathrm{Ker}\,\mathrm{Hom}_\Lambda(T, -)$ are 4, $\begin{smallmatrix}3\\4\end{smallmatrix}$ and 3. Since $\mathrm{Ext}^1_\Lambda\left(\begin{smallmatrix}2\\3\end{smallmatrix}, M\right) \neq 0$ for $M = 4, \begin{smallmatrix}3\\4\end{smallmatrix}$ and $\mathrm{Ext}^2_\Lambda(1,3) \neq 0$, it follows that M is a large partial 2-tilting module. Finally, let $\dot{X}$ be a complex of the form

$$0 \longrightarrow 4 \longrightarrow \begin{matrix}2\\3\\4\end{matrix} \longrightarrow \begin{matrix}1\\2\end{matrix} \longrightarrow 0.$$

Then (iii) clearly holds, and the homology module H of $\dot{X}$ is $H = 3 \oplus 1$. Since $\begin{smallmatrix}2\\3\\4\end{smallmatrix}$ is projective as well as injective, by Lemma 1, we have

(1) $$\mathrm{Hom}_{K(\Lambda)}\left(\dot{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}, \dot{X}[i]\right) = 0 \text{ for every } i \in \mathbb{Z}.$$

Now consider all possible chain maps from $\dot{\begin{smallmatrix}2\\3\end{smallmatrix}}$ to $\dot{X}[i]$. Every picture of the form

(2) $$\begin{array}{ccccccccc} & & 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \longrightarrow & 0 \\ & & & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \\ 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & 0 \end{array}$$

describes a morphism homotopic to zero. On the other hand, for every morphism of the form

$$(3)\qquad \begin{array}{ccccccccc} 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & 0 & & \\ & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & & & \\ 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 \end{array}$$

we have $d = c = 0$. Consequently, by (2) and (3), we obtain

$$(4)\qquad \operatorname{Hom}_{K(\Lambda)}\left(\dot{\begin{smallmatrix}2\\3\end{smallmatrix}}, \dot{X}[i]\right) = 0 \text{ for every } i \in \mathbb{Z}.$$

Finally, we consider all possible maps from $\dot{1}$ to $\dot{X}[i]$. We first note that any picture of the form

$$(5)\qquad \begin{array}{ccccccccccc} & & 0 & \to & \begin{smallmatrix}3\\4\end{smallmatrix} & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 \\ & & & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & & & \\ 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 & & \end{array}$$

describes a morphism homotopic to zero. Moreover, for every morphism of the form

$$(6)\qquad \begin{array}{ccccccccc} 0 & \to & \begin{smallmatrix}3\\4\end{smallmatrix} & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \\ 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 \end{array}$$

we have $c = d = 0$. Putting (5) and (6) together, we get

$$(7)\qquad \operatorname{Hom}_{K(\Lambda)}\left(\dot{1}, \dot{X}[i]\right) = 0 \text{ for every } i \in \mathbb{Z}.$$

Therefore (i) follows from (1), (4) and (7).

Finally, as in (5) any morphism of the form

$$\begin{array}{ccccccccccc} & & 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 \\ & & & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & & & \\ 0 & \to & 4 & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\2\end{smallmatrix} & \to & 0 & & \end{array}$$

is homotopic to zero, and all other nontrivial shifts have zero maps in each degree. Consequently also (ii) holds.

Now, let $\dot{C}$ be an indecomposable complex of the form $0 \longrightarrow \begin{smallmatrix}2\\3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 0$, and let $i : 4 \longrightarrow \begin{smallmatrix}2\\3\\4\end{smallmatrix}$ be the canonical inclusion. Then the picture

$$\begin{array}{ccccccc} 0 & \longrightarrow & 4 & \xrightarrow{\ i\ } & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \longrightarrow & 0 \\ & & {\scriptstyle i}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & \\ 0 & \longrightarrow & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & 0 \end{array}$$

describes a morphism which is not homotopic to zero. Since $\overset{\cdot}{\begin{smallmatrix}2\\3\end{smallmatrix}}$ is a summand of $\dot{T}$, it follows that $\dot{C}$ does not satisfy (i). This remark and the structure of the indecomposable projective modules imply that (iv) holds. ∎

As the following example shows, indecomposable complexes with decomposable components play an important role, even for algebras of global dimension two.

Example C (Left cancellation and left perturbation): There are two nonfaithful large partial 2-tilting modules T and U over a K-algebra Λ with the following properties:

(i) $\mathrm{Hom}_{K(\Lambda)}(\dot{U}, \dot{X}[i]) = 0$ for every $i \in \mathbb{Z}$, where $\dot{X}$ is an indecomposable complex of the form $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$ with P and Q indecomposable projective.

(ii) If $\dot{X}$ satisfies (i), then we have $\mathrm{Hom}_{K(\Lambda)}(\dot{T}, \dot{X}[j]) \neq 0$ for some j.

(iii) $\mathrm{Hom}_{K(\Lambda)}(\dot{T}, \dot{Y}[i]) = 0$ for every $i \in \mathbb{Z}$, for some indecomposable complex $\dot{Y}$ with exactly three non-zero components.

(iv) If $\dot{Y}$ satisfies (iii), then at least one non-zero component of $\dot{Y}$ is decomposable.

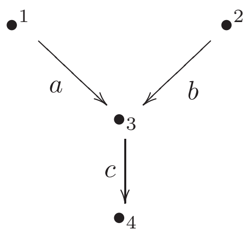
(v) If W is a proper direct summand of T (resp. U), then W is not a large partial 2-tilting module.

(vi) $\dot{X}$ in (ii) and $\dot{Y}$ in (iii) may be complexes $\dot{C} \in K(\Lambda)$ such that

$$\mathrm{Hom}_{K(\Lambda)}(\dot{C}, \dot{C}[j]) = 0$$

for every integer $j \neq 0$.

Construction. Let Λ be the K-algebra given by the quiver



with relations $ca = 0$ and $cb = 0$. Next, let T and U denote the following injective modules:

$$T = 1 \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}, \quad U = \begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}.$$

Then we obviously have $\mathrm{Ext}^2(1,4) \simeq \mathrm{Ext}^1_\Lambda(3,4) \neq 0$ and $\mathrm{Ext}^2\left(\begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix}, 4\right) \simeq \mathrm{Ext}^1_\Lambda(3,4) \neq 0$. Since 4 is the unique indecomposable module in $\mathrm{Ker}\,\mathrm{Hom}_\Lambda(T,-)$ and in $\mathrm{Ker}\,\mathrm{Hom}_\Lambda(U,-)$ and 4 does not lie in $T^{\perp_\infty}$ or $U^{\perp_\infty}$, it follows that T and U are large partial 2-tilting modules. Now, let $\dot{X}$ denote an indecomposable complex of the form

$$0 \longrightarrow \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \longrightarrow 0.$$

Then the following picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} & \xrightarrow{\binom{1}{1}} & \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\ & & 0 & \longrightarrow & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} & \longrightarrow & 0 \end{array} \tag{1}$$

describes a morphism if and only if $\alpha = 0$ and $\beta = 0$. On the other hand, proceeding from left to right, we see that any morphism of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} & \xrightarrow{\binom{1}{1}} & \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & & & \\ 0 & \longrightarrow & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} & \longrightarrow & 0 & & \end{array} \tag{2}$$

is homotopic to zero. Moreover, by Lemma 1, we have $\mathrm{Hom}_{K(\Lambda)}\left(\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}, \dot{X}[i]\right) = 0$ for every $i \in \mathbb{Z}$. By (1) and (2), this implies that $\dot{X}$ satisfies (i). A similar argument clearly works for an indecomposable complex of the form

$$0 \longrightarrow \begin{matrix}3\\4\end{matrix} \longrightarrow \begin{matrix}2\\3\end{matrix} \longrightarrow 0.$$

Next, let $i : 4 \longrightarrow \begin{smallmatrix}3\\4\end{smallmatrix}$ be the canonical inclusion. Then the morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 0} & & & & \\ 0 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 & & \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 0} & & & & \\ 0 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 & & \end{array}$$

are not homotopic to zero. Consequently, since the top complexes are $\dot{1}$ and $\dot{2}$, also (ii) holds.

Finally, let $\dot{Y}$ be a complex of the form

$$0 \longrightarrow 4 \xrightarrow{\alpha} \begin{matrix}3\\4\end{matrix} \oplus \begin{matrix}3\\4\end{matrix} \xrightarrow{\beta} \begin{matrix}1\\3\end{matrix} \oplus \begin{matrix}2\\3\end{matrix} \longrightarrow 0 \ ,$$

where $\alpha = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$. Then the picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}b\\c\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}d\\0\end{smallmatrix}\right)}\downarrow & & \\ 0 & \longrightarrow & 4 & \xrightarrow{\alpha} & \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} & \xrightarrow{\beta} & \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 \end{array} \tag{3}$$

describes a morphism if and only if $\left(\begin{smallmatrix}a\\a\end{smallmatrix}\right) = \left(\begin{smallmatrix}b\\c\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}b\\c\end{smallmatrix}\right) = \left(\begin{smallmatrix}d\\0\end{smallmatrix}\right)$, that is if and only if $a = b = c = d = 0$. Next, let $\dot{\phi}$ be a morphism of the form

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & 4 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 \\ & & & & {\scriptstyle \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}c\\d\end{smallmatrix}\right)}\downarrow & & & & \\ 0 & \longrightarrow & 4 & \xrightarrow{\alpha} & \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} & \xrightarrow{\beta} & \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 & & \end{array} \tag{4}$$

For brevity, let x, y, z, t denote the elements d, $a - b + d$, $b - d$ and $-a + b + c - d$, respectively. Then the morphisms $4 \xrightarrow{z} 4$, $\begin{smallmatrix}3\\4\end{smallmatrix} \xrightarrow{\left(\begin{smallmatrix}y\\x\end{smallmatrix}\right)} \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix} \xrightarrow{\left(\begin{smallmatrix}t\\0\end{smallmatrix}\right)} \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$ tell us that $\dot{\phi} = 0$. Putting (3) and (4) together, we obtain $\operatorname{Hom}_{K(\Lambda)}(\dot{1}, \dot{Y}[i]) = 0$ for every $i \in \mathbb{Z}$. A similar argument shows that $\operatorname{Hom}_{K(\Lambda)}(\dot{2}, \dot{Y}[i]) = 0$ for every i. Moreover, by Lemma 1, we have

$\mathrm{Hom}_{K(\Lambda)}\left(\dot{\begin{smallmatrix}3\\4\end{smallmatrix}}, \dot{Y}[i]\right) = 0$. Thus (iii) holds. On the other hand, the two complexes $0 \longrightarrow 4 \longrightarrow \begin{smallmatrix}3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}i\\3\end{smallmatrix} \longrightarrow 0$ with $i = 1, 2$ are the unique indecomposable complexes with exactly three non-zero and indecomposable components. Therefore (iv) clearly holds. Since T and U have no proper sincere direct summand, also (v) holds. Finally, as in Example A, any morphism of the form

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \big\downarrow & & & & \\
0 & \longrightarrow & \begin{smallmatrix}3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\3\end{smallmatrix} & \longrightarrow & 0 & &
\end{array} \tag{5}
$$

is homotopic to zero. Now, keeping the above hypothesis that $\alpha = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$, let $\dot{\psi}$ denote the morphism

$$
\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & 4 & \xrightarrow{\alpha} & \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} & \xrightarrow{\beta} & \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix}c & d\\e & f\end{smallmatrix}\right)} & & & & \\
0 & \longrightarrow & 4 & \xrightarrow{\alpha} & \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} & \xrightarrow{\beta} & \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & 0 & &
\end{array} \tag{6}
$$

Next, let $\sigma = \left(\begin{smallmatrix}a-d & d\\e & b-e\end{smallmatrix}\right)$, and let $\tau = \left(\begin{smallmatrix}-a+c+d & 0\\0 & -b+e+f\end{smallmatrix}\right)$. Since $\sigma \circ \alpha = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ and $\beta \circ \sigma + \tau \circ \beta = \sigma + \tau = \left(\begin{smallmatrix}c & d\\e & f\end{smallmatrix}\right)$, we get $\dot{\psi} = 0$. Consequently, (vi) follows from (5) and (6). ∎

Note that the complex $\dot{Y}$, constructed in Example C, is big enough to satisfy the following condition

- Every indecomposable projective module occurs as a direct summand of a component of $\dot{Y}$.

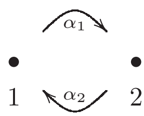
Moreover, every indecomposable projective module over the algebras constructed in Examples A, B and C is a brick. On the other hand, the indecomposable complex $\dot{X}$ constructed in the next example satisfies the following conditions:

- Every indecomposable projective module occurs as a component of $\dot{X}$.
- There exists two indecomposable projective modules, and exactly one of them is (resp., is not) a brick.

Example D (Right addition): There are a non-faithful large partial 2-tilting Λ-module T and an indecomposable complex $\dot{X}$, with exactly 4 non-zero components, satisfying the following conditions:

(i) T is the i-th homology module of $\dot{X}$ for some $i \in \mathbb{Z}$.
(ii) $\mathrm{Hom}_{K(\Lambda)}(\dot{T}, \dot{X}[i]) = 0$ for every $i \in \mathbb{Z}$.
(iii) $\mathrm{Hom}_{K(\Lambda)}(\dot{X}, \dot{X}[j]) = 0$ for every integer $j \neq 0$.
(iv) Let $\dot{C}$ be an indecomposable bounded complex of projective modules such that $\mathrm{Hom}_{K(\Lambda)}(\dot{T}, \dot{C}[i]) = 0$ for every $i \in \mathbb{Z}$. If every non-zero component of $\dot{C}$ is indecomposable, then $\dot{C}$ has exactly four non-zero components.

Construction. Let Λ the Nakajama K-algebra given by the quiver



with relation $\alpha_2\alpha_1 = 0$, considered in [M, Example 3.2] with $n = 2$. Next, let T be the injective module $\begin{smallmatrix}2\\1\end{smallmatrix}$. Then T is a large partial 2-tilting module ([D1, Example 3], a special case of [D2, Example 7]).

Finally, let $\dot{X}$ denote a complex of the form

$$0 \longrightarrow \begin{matrix}1\\2\end{matrix} \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow 0 \ .$$

Then (i) clearly holds. Let $\dot{\varphi}$ be a morphism of the form

$$(1) \qquad \begin{array}{ccccccccccc} & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0. \end{array}$$

Then, proceeding from left to right, we first note that $d \circ \alpha = 0$. Hence, $\mathrm{Ker}\,\beta \neq 0$ and so $d \circ \beta = 0$. This implies that $\mathrm{Ker}\,\gamma \neq 0$. Using these remarks and proceeding from right to left, we conclude that $\dot{\varphi} = 0$. Now, let $\dot{\omega}$ be a morphism of the form

$$(2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \xrightarrow{d} & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow 0. \end{array}$$

Proceeding from right to left, we obtain $\operatorname{Ker}\eta \neq 0$. Hence we have $\operatorname{Ker}\epsilon \neq 0$ and $\delta = 0$. At this point, proceeding from right to left, we see that $\dot{\omega} = 0$. Next, assume $\dot{\tau}$ is a morphism of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 & & \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & & & \\
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots .
\end{array} \tag{3}$$

Since $\operatorname{Ker}\beta \neq 0$, we have $\alpha = 0$. It follows that $\dot{\tau} = 0$. Dually, let $\dot{\sigma}$ be a morphism of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 \; . & &
\end{array} \tag{4}$$

Also in this case, we have $\operatorname{Ker}\beta \neq 0$. Consequently, we get $\dot{\sigma} = 0$. Finally, let $\dot{\rho}$ denote a morphism of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 & & \\
 & & & & {\scriptstyle\rho}\downarrow & & & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots
\end{array}
\quad \left(\text{resp.} \quad
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \cdots \\
 & & & & {\scriptstyle\rho}\downarrow & & & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 \; . & &
\end{array} \right) \tag{5}$$

Since $\rho = 0$ (resp., ρ extends to an endomorphism of the injective module $\begin{smallmatrix}2\\1\\2\end{smallmatrix}$), we have $\dot{\rho} = 0$. Putting (1), (2), (3), (4) and (5) together, we obtain (ii). On the other hand, let $\dot{\lambda}$ be a morphism of the form

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 & & \\
 & & & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 \; .
\end{array} \tag{6}$$

Then the arguments used in (2) show that $\dot{\lambda} = 0$. Consequently, (iii) follows from (1), (3), (4), (5) and (6). Assume now $\alpha : \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ is a non-zero morphism with $\operatorname{Ker}\alpha \neq 0$. Then the morphism

(7)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 & & \\
 & & & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle 0} & & & & \\
 & & 0 & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0\ . & &
\end{array}$$

is not homotopic to zero. Moreover, no morphism of the form

(8)
$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} \longrightarrow \cdots \\
 & & & & \Big\downarrow{\scriptstyle\simeq} & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & 0\ .
\end{array}$$

is homotopic to zero. Since T is a large partial 2-tilting module, (7) and (8) imply that the complexes $\dot{C}$ satisfying (iv) are of the form

(9)
$$0 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow \begin{matrix}2\\1\\2\end{matrix} \longrightarrow 0\ ,$$

and exactly one component is isomorphic to $\begin{smallmatrix}1\\2\end{smallmatrix}$. Let $\beta : \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ be a non-zero morphism. Then the following morphism

(10)
$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\\2\end{smallmatrix} & \longrightarrow & 0\ .
\end{array}$$

is not homotopic to zero. Thus the complex in (9) has less than four components isomorphic to $\begin{smallmatrix}2\\1\\2\end{smallmatrix}$. Therefore, our proof of (ii) shows that also (iv) holds. ∎

The following statement is an immediate consequence of the previous examples.

PROPOSITION 3: *Let V be a large partial n-tilting Λ-module with $\operatorname{pdim}(V) = n$, and let $\dot{C} \in K(\Lambda)$ be an indecomposable complex with the following properties:*
(1) $\operatorname{Hom}_{K(\Lambda)}\left(\dot{V}, \dot{C}[i]\right) = 0$ *for every* $i \in \mathbb{Z}$*;*

(2) *$\dot{C}$ is a bounded partial tilting complex with r non-zero components and s non-zero homology modules.*

(3) *Every indecomposable complex $\dot{D}$ satisfying (1) and (2) has at least s non-zero homology modules.*

Let $H = \bigoplus_{i\in\mathbb{Z}} H_i$, where H_i is the i-th homology module of $\dot{C}$. Then V and $\dot{C}$ may satisfy one of the following conditions:

(a) *n runs over all natural number ≥ 2, $r = s = 2$, H is semisimple but not sincere, and any non-zero component of $\dot{C}$ is indecomposable.*

(b) *$n = s = 2$, $r = 3$, H is semisimple but not sincere, and any non-zero component of $\dot{C}$ is indecomposable (resp., $\dot{C}$ has both decomposable and indecomposable components).*

(c) *$n = s = 2$, $r = 4$, H is sincere but not semisimple, and any non-zero component of $\dot{C}$ is indecomposable.*

(d) *$\dim_K(H) = 2$, and a simple module $S(x)$ is a composition factor of H if and only if $P(x)$ is not a direct summand of V.*

(e) *$\dim_K(H) = 2$, and there is a simple module $S(x)$ such that $S(x)$ is not a composition factor of H and $P(x)$ is not a direct summand of V.*

Proof. (a) Let V denote the module T constructed in Example A (that is the module $\substack{n\\n+1} \oplus \substack{n-2\\n-1} \oplus \cdots \oplus \substack{2\\3} \oplus\ 1$ if n is even, and the module $\substack{n\\n+1} \oplus \substack{n-2\\n-1} \oplus \cdots \oplus \substack{1\\2} \oplus\ 1$ if n is odd). Next, let $\dot{C}$ denote the complex $\dot{X}$ constructed in the same example (of the form $0 \to \substack{2\\3} \to \substack{1\\2} \to 0$ if n is even, and $0 \to \substack{3\\4} \to \substack{2\\3} \to 0$ if n is odd). Then (a) follows from conditions (ii) and (v) in Example A.

(b) Let V denote the module $T = \substack{2\\3\\4} \oplus \substack{2\\3} \oplus\ 1$ constructed in Example B (resp., $T = 1 \oplus 2 \oplus \substack{3\\4}$ constructed in Example C). Next, let $\dot{C}$ denote the complex $\dot{X}$ (resp., $\dot{Y}$) constructed in the same example, of the form

$$0 \to 4 \to \substack{2\\3\\4} \to \substack{1\\2} \to 0 \quad (\text{resp., } 0 \to 4 \xrightarrow{\binom{1}{1}} \substack{3\\4} \oplus \substack{3\\4} \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \substack{1\\3} \oplus \substack{2\\3} \to 0\ ).$$

Then (b) follows from (i) and (ii) in Example B (resp., (iii), (iv) and (vi) in Example C).

(c) Let V denote the module $T = \substack{2\\1}$ constructed in Example D. Next, let $\dot{C}$ denote the complex $\dot{X}$ considered in the same example, of the form $0 \to \substack{1\\2} \to \substack{2\\1\\2} \to \substack{2\\1\\2} \to \substack{2\\1\\2} \to 0$. Then (c) follows from (ii) and (iii) in Example D.

(d) This is an immediate consequence of the above proof of (a) with $n = 2$, $V = {}^{2}_{3} \oplus 1$ and $\dot{C} = \dot{X}$ of the form $0 \to {}^{2}_{3} \to {}^{1}_{2} \to 0$.

(e) For every $n \geq 5$, it suffices to choose V and $\dot{C}$ as in the proof of (a) and $x = 5$ if n is even (resp., $x = 6$ if n is odd). ■

We end with three remarks concerning the presence/absence of

- Classes of modules in several definitions of tilting-type modules (Remark 4);
- non-zero and non surjective morphisms in the construction of several complexes (Remark 5).
- Enough injective summands of large partial tilting modules (Remark 6).

Remark 4: In this paper we use only the "discrete" definition of n-tilting and partial n-tilting modules, recalled at the beginning of Section 2. We refer to [B, Proposition 3.6 and Lemma 3.12] for "continuous" characterization of these modules (and their duals), by means of two classes of modules. By repeating the words of [R3, Section II, Module Theory], we recall that many "subcategories of module categories" play an important role in Tilting Theory. In some sense, our results on bounded complexes describe an "implicit finiteness property" [KT, Introduction] of tilting modules. (For the relationship between modules "of finite type" and tilting modules, see, for instance, [AnHT, BH, StT, BSt] and the other papers quoted in [T, Section 4]).

Remark 5: In some of the examples used to prove Proposition 3, the relationship between V and $\dot{C}$ is explained by a morphism of the form $f : L \longrightarrow M$ with the following properties:

- L is an indecomposable summand of V.
- M is an indecomposable projective module, such that $M \in V^{\perp_\infty}$ but $M \notin \mathrm{add}(V)$.
- The left hand side of $\dot{C}$ describes the complex $\dot{L}$, while the right hand side describes M.

We list in the sequel some examples of this kind.

(1) $L = {}^{\ i}_{i+1}$, $M = {}^{i-1}_{\ i}$ with $i = 2$ if n is even (resp., $i = 3$ if n is odd) and $\dot{C}$ is a complex of the form $0 \to {}^{\ i}_{i+1} \to {}^{i-1}_{\ i} \to 0$ (see Example A(ii) and Proposition 3(a)).

(2) $L = {}^{2}_{3}$, $M = {}^{1}_{2}$ and $\dot{C}$ is a complex of the form $0 \to 4 \to {}^{2}_{\substack{3\\4}} \to {}^{1}_{2} \to 0$ (see Example B and Proposition 3(b)).

(3) $L = \substack{2\\1}$, $M = \substack{2\\1\\2}$ and $\dot{C}$ is a complex of the form $0 \to \substack{1\\2} \to \substack{2\\1\\2} \to \substack{2\\1\\2} \to \substack{2\\1\\2} \to 0$ (see Example D and Proposition 3(c)).

Remark 6: The construction of the complex $\dot{Y}$, satisfying condition (iii) of Example C, makes use of a decomposable complex, that is of the projective resolution $\dot{B}$ of the decomposable injective summand $B = 1 \oplus 2$ of the given partial 2-tilting module $T = 1 \oplus 2 \oplus \substack{3\\4}$. In this case we obtain the complex $\dot{Y}$ from $\dot{B}$ by replacing the first non-zero map on the left, that is the inclusion $4 \oplus 4 \longrightarrow \substack{3\\4} \oplus \substack{3\\4}$, with the "diagonal" map $\binom{1}{1} : 4 \longrightarrow \substack{3\\4} \oplus \substack{3\\4}$.

One can use Example C also to see that the useful complexes $\dot{B}$ (involved in the four operations of Section 2) are not necessarily direct summands of the given partial tilting complex. For instance, we obtain the complex $\dot{X}$ satisfying condition (ii) of Example C from the projective resolution $\dot{B}$ of the simple injective module $B = 1$. In this case, B is not a direct summand of the given partial tilting module $U = \begin{smallmatrix}1&&2\\&3&\end{smallmatrix} \oplus \substack{3\\4}$. However, the complex $\dot{U} \oplus \dot{B}$, as well as $\dot{U}$, is again a partial tilting complex, which is not a tilting complex.

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